Institute for Advancing Intelligence (IAI), TCG CREST
Mid-Semestral Examination
Ph.D Program Session: 2020–2021
Discrete Mathematics

Date: 18.10.2020 Marks: 60 Time: 4 Hours

Answer as much as you can. Total marks is 80 and the maximum you can score is 60.

1. (a) There are two tribes of trolls, the Glums and the Plogs. Glums are all truthtellers and Plogs are all liars. You meet two trolls one day, Kim and Coin. Kim says "*We are from different clans.*" Coin says "*Kim is a liar.*" Which tribe is Kim from and which tribe is Coin from?

   (b) During a murder investigation, you have gathered the following clues:

   - If the knife is in the store room, then we see it when we clear the store room,
   - The murder was committed at the basement or inside the apartment,
   - If the murder was committed at the basement, then the knife is in the dust bin,
   - We did not see a knife when we cleared the store room,
   - If the murder was committed outside the building, then we are unable to find the knife,
   - If the murder was committed inside the apartment, then the knife is in the store room.

   Can you deduce "*where the knife is*"?

   (c) A function $f : \{0,1\}^n \rightarrow \{0,1\}$ is called "*self-dual*" if $f(x_1 x_2 \cdots x_n) = f(\overline{x_1}\ \overline{x_2} \cdots \overline{x_n})$, for all $x_1 x_2 \cdots x_n$, where $\overline{x_i}$ is complement of the bit x_i. Compute the number of self dual functions that are present over n tuples. **[3+4+3=10]**

2. (a) A 4×4 S-box is a permutation that takes 4-bit input and produces a 4-bit output. An S-box is called "*good*" if the S-box contains no fixed points (i.e. $\forall x \in \{0,1\}^4,\ S(x) \neq x$). Count the number of 4×4 *good* S-boxes.

(b) You are presented with 2 fuses, each of which will burn for exactly 1 minute, but not uniformly along its length. Can you measure 45 seconds by this 2 fuses? Find out the set of all possible times which can be measured by using n fuses. [Hint: You can burn both the sides of a fuse, if required.] **[5+5=10]**

3. (a) You are kept in a prison consisting of n^2 cells arranged like the squares of an $n \times n$ chessboard. There are doors between all adjoining cells. You are in one of the corner cells and it is told that you can get out of the prison provided you can get into the diagonally opposite corner cell after passing through *every other cell exactly once*. Can you obtain freedom?

   (b) Inspector Bob has to put some criminals in cells of the prison. The criminals are notorious and can beat one another to death. If any criminal dies inside the cell, then the inspector will lose his job. In this scenario, the inspector thought of putting each criminal in a cell. But, his boss wants it to be done using the minimum number of cells. The only saving grace for the inspector is that the criminals fight according to the following pattern: (i) a criminal does not beat himself, (ii) if a criminal C_1 does not beat a criminal C_2, and criminal C_2 does not beat criminal C_3, then criminal C_1 does not beat criminal C_3 and vice-versa. Help the inspector by solving this problem efficiently. **[5+5=10]**

4. (a) In how many ways can you distribute k identical pieces of candy to n children such that each child gets at least j pieces?

   (b) A committee of 5 is to be chosen from a club that boast a membership of 12 men and 10 women. How many ways can the committee be formed if it has to have at least 2 women? How many ways if, in addition, one particular man and one particular woman who are members of the club, refuse to serve together on the committee? **[4+6=10]**

5. (a) There are two candidates A and B are elected for vote. After the election is over, it was checked that both of them receives n votes. In how many ways the voting can happen such that at any point of time, the number of votes obtained by A is at least the number of votes obtained by B ?

   (b) How can you select the winner of a lottery (among 20 people) uniformly at random using an unbiased coin? **[5+5=10]**

6. (a) Prove that any positive integer a can be uniquely written as $a = 2^k q$ for some integer $k \geq 0$ and q is an odd integer.

   (b) Show that if $(n+1)$ integers are chosen from the set $\{1, 2, 3, \ldots, 2n\}$ then one of the chosen integers divides another.

   (c) The general of a battelion A, let G_A, wants to inform the number of soldiers he has to the general of battelion B, let G_B. But G_A also wants to make sure that the enemy should not know the number of soldiers he has. For this, G_A came up with an idea. He asks his soldiers to line up in in rows of 11, then in rows of 17, 29, and 31. Respectively, each time, he noted down with remainder $8, 5, 16$, and 24. G_A passes the information $(8, 11), (5, 17), (16, 29)$ and $(24, 31)$. Can you say whether G_B can deduce the number of soldiers G_A has from the passed on information. If yes, then how many soldiers G_A has? **[2+3+5=10]**

7. (a) Prove that for all non-zero positive integers n,

$$F_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-i-1}{i},$$

   where F_n is the n-th fibonacci number.

   (b) We define a set $\mathcal{S} \subset \mathbb{N}$, such that $|\mathcal{S}|$ is finite, to be "*crazy*" if $|\mathcal{S}| \in \mathcal{S}$. How many subsets of $\{1, 2, \ldots, n\}$ are there that are minimal crazy sets (minimality is defined in the sense that subsets that are crazy and do not properly contain any other crazy set). E.g., for a set $\{1, 2, 3\}$, the minimal crazy sets are $\{1\}$ and $\{2, 3\}$.

   (c) Find the minimal crazy subsets of the set $\{1, 2, 3, 4, 5\}$.

[4+4+2=10]

8. (a) What is the generating function for $\{a\}_k$ where a_k is the number of solutions of $x_1 + x_2 + x_3 + x_4 = k$, when x_1, x_2, x_3 and x_4 are integers with $x_1 \geq 3, 1 \leq x_2 \leq 5, 0 \leq x_3 \leq 4$ and $x_4 \geq 1$.

   (b) You flip three fair coins (i.e., probability of head = probability of tail = 1/2). At least two of the outcomes will be identical, and it is an even chance that the third outcome is a tail or a head. Therefore, Pr[all are alike] = 1/2. Do you agree with this argument ? Justify your answer.

   (c) There are three coins in a box. One is a two-headed coin, another is a fair coin (i.e., probability of head = probability of tail = 1/2) and the third one is a biased coin with probability of head = 3/4. You choose one of the three coins at random and flipped, it shows head. What is the probability that it is the two-headed coin? [**4+3+3=10**]