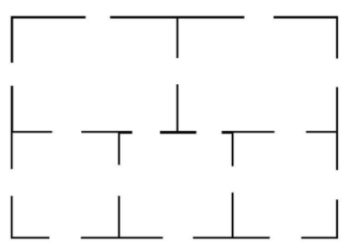
Institute for Advancing Intelligence (IAI), TCG CREST
Semester Examination
Ph.D Program Session: 2020–2021
Discrete Mathematics

Date: 24.12.2020 Marks: 100 Time: 5 Hrs.

Answer as much as you can. The total Marks is 130, and the maximum you can obtain is 100.

1. (a) Consider the following floor plan of five room research lab at IAI in Figure 1. Show that you can not find a continuous path that pass through each door exactly once. Now if you are allowed to close some doors of the lab, after closing at least how many doors you will find a continuous path that passes through each door exactly once?

Figure 1: Floor Plan of IAI Research Lab



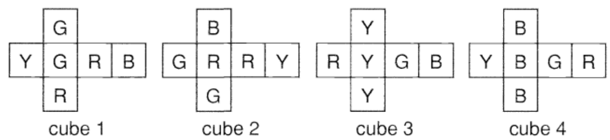
   (b) Consider the king in a 7×7 chessboard which can move to horizontal and vertical adjacent squares. Show that you can not have a *king's* tour that traverse exactly 48 other cells, and comes back to the starting square. Can you conclude the same for an 8×8 chessboard?

   (c) Can you tile a 4×3 checkerboard with dominoes (a domino being two adjacent squares)? Formulate this problem as a graph theoretic problem, and find the solution. **[8+6+6=20]**

2. (a) Given four cubes whose faces are colored (R)ed, (B)lue, (G)reen and (Y)ellow, as in the Figure 2, can you pile them up so that all

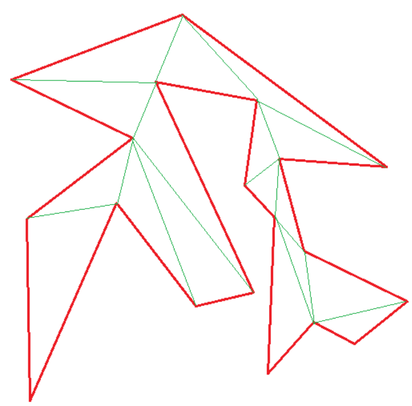
four colours appear on each side of the resulting 4×1 stack? If yes, show one such as configuration.

Figure 2: Four Cubes



(b) You are the owner of a museum whose structure is depicted in Figure 3. What is the minimum number of CCTV cameras (also find the locations) that you will set up to ensure security of the interior of the museum? Assume that you can set up the CCTV camera only at the corners.

Figure 3: Structure of the Museum



(c) Robot Sophia is walking on a *cyclic track*. The track is marked at evenly spaced intervals with 0s and 1s, with a total of 16 marks. Sophia can see the 4 marks closest to her. How should the 0s and 1s be put on the track so that she knows where on the track she is by just looking at the 4 closest marks? **[6+6+8=20]**

3. (a) There are 10 teams in the ISL tournament. In each round the teams are paired and they play each other once. Prove that after 4 rounds, there are three teams who have not played against each other. [Hint: Try to prove and use the following fact: $K_{n,n}$ is the only n-regular graph with $2n$ vertices that does not contain any 3-cycle.]

   (b) In a group of 100 people, each one knows at least 67 other people. Prove that there exist 4 people who are mutual friends. **[9+6=15]**

4. Proof or Refute with brief argument.

   (a) G is k-connected ($k \geq 2$) if and only if any set of k vertices is contained in a cycle.

   (b) Suppose an undirected graph G with unique positive weights has a minimum spanning tree T. If we square all the edge weights and compute the minimum spanning tree again, we will get the same tree structure again.

   (c) We can sort the vertices of a di-graph topologically if and only if the graph is acyclic. (Hint: In an acyclic digraph, there exist at least one source and at least one sink)

   (d) In a group of 3^{10} lords, there is either a 10-enemies set or a 10-alliance set. **[3+3+5+4=15]**

5. A tree T is called an $(1,3)$-tree if all of it's vertices have degree either 1 or 3.

   (a) How many $(1,3)$-trees are there with the degree sequence $(3,3,3,3,1,1,1,1)$?

   (b) Draw an $(1,3)$-tree with 19 vertices.

   (c) If T has n leaves, find the number of vertices with degree 3.

   (d) Let T has $n \geq 4$ vertices. Show that there is some internal vertex which is adjacent to exactly two leaves.

   (e) Find the number of $(1,3)$-trees with n leaves. **[2+3+3+4+3=15]**

6. (a) Show that, any connected planar graph with n-vertices contains an independent set of size $n/4$.

   (b) Prove that a k-regular graph with smallest cycle of length 5 has at least k^2+1 vertices.

(c) Let G be a simple graph with 19 edges, and degree of each vertex is greater than 3. Knowing nothing else about G, find (i) the maximum number of vertices that G could have, (ii) the maximum number of vertices that G could have for which one can conclude whether G is planar or not. **[4+5+6=15]**

7. (a) An $m \times n$ integer matrix M is called a *m*agic rectangle if it satisfies the following two properties:

   - $1 \leq M_{ij} \leq n$
   - No two entries in any row or in any column are equal.

   The above two conditions immediately imply that $m \leq n$. For example, consider the following matrix which is a 3×5 magic rectangle

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 5 & 2 & 1 & 4 \end{bmatrix}$$

   Show that it is always possible to turn a magic rectangle to a magic square by inserting additional rows that satisfies the above two properties. (Hint: Convert the problem to a matching problem)

   (b) If the cardinality of a maximum independent set of a tree T with n vertices is k, then find the size of its maximum matching in terms of n and k.

   (c) We know that if G is a bipartite graph with no isolated vertices, then $\alpha(G) = \beta'(G)$. Show that a graph is bipartite if and only for every subgraph H of G with no isolated vertices $\alpha(H) = \beta'(H)$ holds. **[7+3+5=15]**

8. (a) Given a tree T with n vertices, design an algorithm to color the edges of T using optimal number of colors.

   (b) A map is a 3-connected planar graph such that it does not contain any cutset with 1 or 2 edges and in particular no vertex of degree 1 or 2. A map is said to be k-colorable if its faces can be colored with k colors so that no two faces with a boundary edge in common have the same color. Prove that a map G is 2-colorable if and only if G is Eulerian. **[8+7=15]**